

# Boundary Control method and De Branges spaces. Schrödinger equation, Dirac system and Discrete Schrödinger operator.

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**Abstract.** In the framework of the application of the Boundary Control method to solving the inverse dynamical problems for the one-dimensional Schrödinger and Dirac operators on the half-line and semi-infinite discrete Schrödinger operator, we establish the connections with the method of De Branges: for each of the system we construct the De Branges space and give a natural dynamical interpretation of all its ingredients: the set of function the De Brange space consists of, the scalar product, the reproducing kernel.

## 1. Introduction

In [2, 6] the authors attempted to look at different approaches to inverse problems for one-dimensional systems from one (dynamical) point of view. It happens that Gelfand-Levitan [10], Krein [12], Simon [20] and Remling [17] equations can be derived within the framework of the Boundary Control method. At the same time all the ingredients of corresponding equations has their dynamical counterparts.

In [17, 18] the author answering the questions posed by Simon in [20, 11], used the De Branges method and De Branges spaces. In [2] the authors have shown that the equations derived by Remling [18] are in fact Krein equations and they have clear dynamical interpretation. In the present paper we would like to elaborate this observation: in fact the link between Boundary Control method and De Branges method are much deeper. In our approach we deal with the dynamical systems with boundary control. Fixing time  $T$  we take the set of the states of the system at this time (the reachable set), taking the Fourier image of this set we get the new space. We equip this space with the norm generated by so-called *connecting operator* to get a Hilbert space of analytic functions. Then we construct the reproducing kernel in this space by solving the Krein equation. We develop this approach on the basis of three systems: Schrodinger equation and Dirac system on the half-line and semi-infinite discrete Schrodinger operator.

In the second section we provide all necessary information on De Branges spaces following to [18] and [19]. In the third section we deal with the Schrödinger operator on the half-line, the forth and fifth sections are devoted to the Dirac operator on

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the half line and the semi-infinite discrete Schrödinger operator. For each operator we consider the dynamical settings of the inverse problem, introduce the dynamical inverse data and operators of the BC method. Then for each dynamical problem we introduce special spaces which (as we prove) will be the De Branges spaces.

## 2. De Branges spaces

Here we provide the information on De Branges spaces according to [18, 19]. We call entire function  $E : \mathbb{C} \mapsto \mathbb{C}$  a *Hermite-Biehler function* if  $|E(z)| > |E(\bar{z})|$  for  $z \in \mathbb{C}_+$ . Let  $F^\#(z) = \overline{F(\bar{z})}$ . The Hardy space  $H_2$  is defined by:  $f \in H_2$  if  $f$  is holomorphic in  $\mathbb{C}_+$  and  $\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx < \infty$ . Then De Branges space  $B(E)$  consists of entire functions such that:

$$B(E) := \left\{ F : \mathbb{C} \mapsto \mathbb{C}, F \text{ entire, } \int_{\mathbb{R}} \left| \frac{F(\lambda)}{E(\lambda)} \right|^2 d\lambda < \infty, \frac{F}{E}, \frac{F^\#}{E} \in H_2 \right\}.$$

The space  $B(E)$  with the scalar product

$$[F, G]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} \overline{F(\lambda)} G(\lambda) \frac{d\lambda}{|E(\lambda)|^2}$$

is a Hilbert space. For any  $z \in \mathbb{C}$  the *reproducing kernel* is introduced by

$$(2.1) \quad J_z(\xi) := \frac{\overline{E(z)}E(\xi) - E(\bar{z})\overline{E(\bar{\xi})}}{2i(\bar{z} - \xi)}$$

Then

$$F(z) = [J_z, F]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} \overline{J_z(\lambda)} G(\lambda) \frac{d\lambda}{|E(\lambda)|^2}$$

We observe that a Hermite-Biehler function  $E(\lambda)$  defines  $J_z$  by (2.1). The converse is also true [9, 8]:

**THEOREM 1.** *Let  $X$  be a Hilbert space of entire functions with reproducing kernel such that*

- 1) *For any  $\omega \in \mathbb{C}$  point evaluation is a bounded functional, i.e.  $|f(\omega)| \leq C\|f\|_X$ .*
- 2) *if  $f \in X$  then  $f^\# \in X$  and  $\|f\|_X = \|f^\#\|_X$*
- 3) *if  $f \in X$  and  $\omega \in \mathbb{C}$  such that  $f(\omega) = 0$ , then  $\frac{z-\bar{\omega}}{z-\omega}f(z) \in X$  and  $\left\| \frac{z-\bar{\omega}}{z-\omega}f(z) \right\|_X = \|f\|_X$ .*

*then  $X$  is a De Branges space based on the function*

$$E(z) = \sqrt{\pi}(1-iz)J_i(z)\|J_i\|_X^{-1}.$$

*where  $J_z$  is a reproducing kernel.*

## 3. Schrödinger equation on the half-line

For the potential  $q \in L_{1,loc}(\mathbb{R}_+)$  we consider the Schrödinger operator on the half-line  $H = -\partial_x^2 + q$  on  $L_2(0, \infty)$  with Dirichlet boundary condition  $\phi(0) = 0$ . For  $z \in \mathbb{C}$  consider the solution

$$(3.1) \quad \begin{cases} -\varphi''(x) + q(x)\varphi(x) = z\varphi(x), \\ \varphi(0, z) = 0, \quad \varphi(0, z) = 1. \end{cases}$$

We fix  $N \in \mathbb{R}_+$  and show that the function  $E(z) := \varphi(N, z) + i\varphi'(N, z)$  is a Hermite-Biehler function. First we observe that  $\varphi(N, \bar{z}) = \overline{\varphi(N, z)}$  and  $\varphi'(N, \bar{z}) = \overline{\varphi'(N, z)}$ , and consider (2.1):

$$\begin{aligned}
 J_z(\xi) &= \frac{\left(\overline{\varphi(z)} - i\overline{\varphi'(z)}\right) (\varphi(\xi) + i\varphi'(\xi)) - (\varphi(\bar{z}) + i\varphi'(\bar{z})) \overline{(\varphi(\xi) + i\varphi'(\xi))}}{2i(\bar{z} - \xi)} \\
 &= \frac{\left(\overline{\varphi(z)} - i\overline{\varphi'(z)}\right) (\varphi(\xi) + i\varphi'(\xi)) - \left(\overline{\varphi(z)} + i\overline{\varphi'(z)}\right) (\varphi(\xi) - i\varphi'(\xi))}{2i(\bar{z} - \xi)} \\
 (3.2) \qquad \qquad \qquad &= \frac{-\overline{\varphi'(z)}\varphi(\xi) + \overline{\varphi(z)}\varphi'(\xi)}{\bar{z} - \xi}
 \end{aligned}$$

We take two points  $z, \xi \in \mathbb{C}$  and consider

$$\begin{aligned}
 -\overline{\varphi''(x)} + q(x)\overline{\varphi(x)} &= \bar{z}\overline{\varphi(x)}, \\
 -\varphi''(x) + q(x)\varphi(x) &= \xi\varphi(x)
 \end{aligned}$$

multiply the first equation by  $u(\xi)$ , multiply the second by  $\bar{u}(z)$  and subtract to get

$$-\overline{\varphi''(z)}\varphi(\xi) + \varphi''(\xi)\overline{\varphi(z)} = (\bar{z} - \xi)\overline{\varphi(z)}\varphi(\xi)$$

We integrate the above equality from 0 to  $N$  and integrate by parts to get :

$$\begin{aligned}
 (\bar{z} - \xi) \int_0^N \overline{\varphi(x, z)}\varphi(x, \xi) dx &= \left[ -\overline{\varphi'(x, z)}\varphi(x, \xi) + \varphi'(x, \xi)\overline{\varphi(x, z)} \right] \Big|_{x=0}^{x=N} \\
 (3.3) \qquad \qquad \qquad &= -\overline{\varphi'(N, z)}\varphi(N, \xi) + \varphi'(N, \xi)\overline{\varphi(N, z)}
 \end{aligned}$$

Comparing (3.2) and (3.3) we see that

$$J_z(\xi) = \int_0^N \overline{\varphi(x, z)}\varphi(x, \xi) dx$$

Taking  $\xi = z$ :

$$(3.4) \quad 0 < \int_0^N \|\varphi(x, z)\|^2 dx = J_z(z) = \frac{|E(z)|^2 - |E(\bar{z})|^2}{2i(-2\operatorname{Im} z)} = \frac{|E(z)|^2 - |E(\bar{z})|^2}{4\operatorname{Im} z}$$

which proves  $E$  to be a Hermite-Biehler function. Thus one can define the De Branges space  $\widehat{B}_S^N$  based on the function  $E$ . The De Branges theory [8] says that every De Branges space corresponds to a certain canonical system, and provides the procedure of recovering this system from the space (essentially from the function  $E$ ). So, once one have in hands the space  $\widehat{B}_S^N$ , and we know that this space comes from Schrödinger equation (special case of canonical system), it is reasonable to pose question of recovering the potential  $q$  (see [17, 18]). Below we construct the De Branges space of Schrödinger operator using the dynamical approach. And it will be explained that "inverse problem", i.e. recovering of the system from the De Branges space is equivalent to Boundary Control method [5, 2, 6].

It is known [13] that there exist a spectral measure  $d\rho(\lambda)$ , such that for all  $f, g \in L^2(\mathbb{R}_+)$  the Parseval identity holds:

$$(3.5) \quad \int_0^\infty f(x)g(x) dx = \int_{-\infty}^\infty (Ff)(\lambda)(Fg)(\lambda) d\rho(\lambda),$$

where  $F : L_2(\mathbb{R}_+) \mapsto L_{2,\rho}(\mathbb{R})$  is a Fourier transformation:

$$(3.6) \quad \begin{aligned} (Ff)(\lambda) &= \int_0^\infty f(x)\varphi(x, \lambda) dx \\ f(x) &= \int_{-\infty}^\infty (Ff)(\lambda)\varphi(x, \lambda) d\rho(\lambda). \end{aligned}$$

For the same potential  $q$  we consider the initial boundary value problem for the 1d wave equation on the half line:

$$(3.7) \quad \begin{cases} u_{tt}(x, t) - u_{xx}(x, t) + q(x)u(x, t) = 0, & x > 0, t > 0, \\ u(x, 0) = u_t(x, 0) = 0, & u(0, t) = f(t). \end{cases}$$

where  $f$  is an arbitrary  $L_{loc}^2(\mathbb{R}_+, \mathbb{C})$  function referred to as a *boundary control*. The following representation [2, 6] for  $u^f$  holds:

$$(3.8) \quad u^f(x, t) = \begin{cases} f(t-x) + \int_x^t w(x, s)f(t-s) ds, & x \leq t, \\ 0, & x > t. \end{cases}$$

Let  $\mathcal{F}^T := L^2(0, T; \mathbb{C})$  with the scalar product  $(f, g)_{\mathcal{F}^T} = \int_0^T \overline{f(t)}g(t) dt$  be the outer space, the *space of controls*. The dynamical Dirichlet-to-Neumann map  $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  for the system (3.7) is defined by

$$(3.9) \quad (R^T f)(t) = u_x^f(0, t), \quad t \in (0, T),$$

with the domain  $\{f \in C^2([0, T]; \mathbb{C}) : f(0) = f'(0) = 0\}$ . According to (3.8) it has a representation

$$(3.10) \quad (R^T f)(t) = -f'(t) + \int_0^t r(s)f(t-s) ds, \quad r(s) = w_x(0, s).$$

The wave, generated by (3.7) propagate with unite velocity, that is why the natural setting of the dynamical inverse problem [2, 6] is to recover  $q(x)$ ,  $x \in (0, T)$  from  $R^{2T}$ , or what is equivalent, from  $r(t)$ ,  $t \in (0, 2T)$ .

Introduce the inner space, the *space of states*  $\mathcal{H}^T = L_2(0, T; \mathbb{C})$  with the scalar product  $(a, b)_{\mathcal{H}^T} = \int_0^T \overline{a(t)}b(t) dt$  and a *control operator*

$$W^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \quad W^T f := u^f(\cdot, T).$$

Notice that for the equation (3.7) it is natural to consider [2, 6] the real controls (and, consequently, the real space of states), but all the results are trivially generalized to the case of complex  $\mathcal{F}^T$ ,  $\mathcal{H}^T$ . Everywhere below, unless it is mentioned, we consider only real controls. The following statement is valid:

**THEOREM 2.** *Control operator  $W^T$  is an isomorphism.*

The solution  $u^f$  to (3.7) admits the spectral representation [2] at fixed time  $T$ :

$$(3.11) \quad W^T f := u^f(x, T) = \int_{-\infty}^\infty \int_0^T \frac{\sin \sqrt{\lambda}s}{\sqrt{\lambda}} f(T-s) ds \varphi(x, \lambda) d\rho(\lambda)$$

We take a Fourier transform (3.6) of a state  $u^f(\cdot, T)$ , generated by a control  $f$ , for  $\mu \in \mathbb{R}$  we get:

$$(3.12) \quad \begin{aligned} \widehat{u}^f(\mu, T) &= \int_{-\infty}^\infty u^f(x, T) \varphi(x, \mu) dx \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^T \frac{\sin \sqrt{\lambda}s}{\sqrt{\lambda}} f(T-s) ds \varphi(x, \lambda) d\rho(\lambda) \varphi(x, \mu) dx = \int_0^T \frac{\sin \sqrt{\mu}s}{\sqrt{\mu}} f(T-s) ds. \end{aligned}$$

Since  $\int_{-\infty}^{\infty} u^f(x, T) \varphi(x, \mu) dx$  is analytic in  $\mathbb{C}$ , we can continue  $u^f(\mu, T)$  from  $\mathbb{R}$ , to get

$$(3.13) \quad \widehat{u}^f(\mu, T) = \int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} f(T-s) ds, \quad \mu \in \mathbb{C}.$$

We introduce the space of the Fourier images of states of the dynamical system (3.7) at time  $T$  (controls are real here):

$$B_S^T := \{ \widehat{u}^f(\mu, T) \mid f \in \mathcal{F}^T \}.$$

Which we put as a definition of De Brange space. Bearing in mind (3.13), we get

$$(3.14) \quad B_S^T = \left\{ \int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} f(T-s) ds \mid f \in \mathcal{F}^T \right\}$$

In [17, 18], the author have shown that  $B_S^T$  (precisely (3.14)) is a De Branges space. Our aim will be to show the same using the dynamical approach.

We introduce the *connecting operator*  $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  using the quadratic form:

$$(3.15) \quad (C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}^T}, \quad C^T = (W^T)^* W^T.$$

The connecting operator is an isomorphism in  $\mathcal{F}^T$ , [2, 6].

We can evaluate using the Parseval identity (3.5) and definition of  $C^T$ :

$$(C^T f, g)_{\mathcal{F}^T} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}^T} = \int_{-\infty}^{\infty} \overline{\widehat{u}^f(\mu, T)} \widehat{u}^g(\mu, T) d\rho(\mu)$$

then we use (3.12), which yields:

$$(3.16) \quad \begin{aligned} (C^T f, g)_{\mathcal{F}^T} &= \int_{-\infty}^{\infty} \overline{\int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} f(T-s) ds} \int_0^T \frac{\sin \sqrt{\mu} t}{\sqrt{\mu}} g(T-t) dt d\rho(\mu) \\ &= \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho(\mu), \end{aligned}$$

$$(3.17) \quad F(\mu) = \int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} f(T-s) ds, \quad G(\mu) = \int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} g(T-s) ds.$$

Then for the functions  $F, G \in B_S^T$  having the representations (3.17), we can introduce the scalar product in  $B_S^T$  by

$$(3.18) \quad [F, G]_{B_S^T} := (C^T f, g)_{\mathcal{F}^T}.$$

The fact that  $C^T$  is an isomorphism implies that the space  $B_S^T$  equipped with the norm, generated by this scalar product is a Hilbert space.

For positive  $N$  we can prescribe a self-adjoint boundary condition at  $x = N$  :

$$(3.19) \quad \begin{cases} -\varphi''(x) + q(x)\varphi(x) = z\varphi(x), \\ \varphi(0, z) = 0, \quad \alpha\varphi(N, z) + \beta\varphi'(N, z) = 0. \end{cases}$$

The (discrete) measure corresponding to (3.19) we denote by  $d\rho_N(\lambda)$ .

REMARK 1. *Due to the finite speed of wave propagation in the dynamical system (3.7), equal to one, in all formulaes starting from (3.11), we can substitute the measure  $d\rho(\lambda)$  by any measure  $d\rho_N(\lambda)$  with  $N \geq T$ . And consequently,*

$$[F, G]_{B_S^T} = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho(\mu) = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho_N(\mu).$$

It is a crucial fact in BC method that  $C^T$  admits the representation in terms of the inverse data [2, 6]:

**THEOREM 3.** *Control operator  $C^T$  admits the representation in terms of the dynamical data*

$$(3.20) \quad (C^T f)(t) = f(t) + \int_0^T c^T(t, s) f(s) ds, \quad 0 < t < T,$$

where

$$(3.21) \quad c^T(t, s) = [p(2T - t - s) - p(t - s)], \quad p(t) := \frac{1}{2} \int_0^{|t|} r(s) ds$$

and spectral data:

$$(C^T f)(x) = \int_0^T C(x, y) f(y) dy, \quad C(x, y) = \int_{-\infty}^{\infty} \frac{\sin \sqrt{\mu}(T - x)}{\sqrt{\mu}} \frac{\sin \sqrt{\mu}(T - y)}{\sqrt{\mu}} d\rho(\lambda),$$

where the action of generalized kernel  $C(x, y)$  is defined by (3.16).

Let  $J_z$  be the reproducing kernel in  $B_S^T$ , the latter means that for all  $F \in B_S^T$  the following should hold for  $z \in \mathbb{C}$ :

$$(3.22) \quad [J_z, F]_{B_S^T} = F(z).$$

Let  $F(\mu) = \int_0^T \frac{\sin \sqrt{\mu}s}{\sqrt{\mu}} f(T - s) ds$ . We look for  $J_z$  in the form:

$$(3.23) \quad J_z(\mu) = \int_0^T \frac{\sin \sqrt{\mu}(T - s)}{\sqrt{\mu}} j_z(s) ds.$$

Evaluating l.h.s. of (3.22) using (3.18) and r.h.s. of (3.22) using representation of  $F$  and fact that  $f$  is real, we arrive at

$$(C^T j_z, f)_{\mathcal{F}^T} = \int_0^T \frac{\sin \sqrt{z}(T - s)}{\sqrt{z}} f(s) ds,$$

which immediately yields the following Krein equation on  $j_z$ :

$$(3.24) \quad (C^T j_z)(s) = \frac{\sin \sqrt{z}(T - s)}{\sqrt{z}}, \quad s \in (0, T).$$

Notice that (3.24) has a unique solution due to the fact that  $C^T$  is an isomorphism.

Let us set up the *special control problem*: for  $z \in \mathbb{C}$  to find a (complex-valued) control  $f_z \in L_2(0, T; \mathbb{C})$  such that  $W^T f_z = \varphi(x, z)$ ,  $x \in (0, T)$ . Notice that only here we deal with complex-valued controls.

**LEMMA 1.** *The solution of the special control problem can be found as a unique solution to the Krein equation (3.24).*

**PROOF.** We take the equality

$$W^T f_z = \varphi(x, z), \quad x \in (0, T),$$

and multiply it in  $\mathcal{H}^T$  by  $W^T g$ ,  $g \in \mathcal{F}^T$ . As result we get that

$$(3.25) \quad (W^T f_z, W^T g)_{\mathcal{H}^T} = (\varphi(\cdot, z), W^T g)_{\mathcal{H}^T} = \overline{\int_0^T (W^T g)(x) \varphi(x, z) dx}.$$

The definition of  $C^T$  (3.15) and spectral representation (3.11) transform (3.25) to:

$$(3.26) \quad (C^T f_z, g)_{\mathcal{F}^T} = \overline{\int_0^T \int_{-\infty}^{\infty} \int_0^T \frac{\sin \sqrt{\lambda} s}{\sqrt{\lambda}} g(T-s) ds \varphi(x, \lambda) d\rho(\lambda) \varphi(x, z) dx},$$

From (3.26) as in the proof of (3.12) we deduce that

$$(C^T f_z, g)_{\mathcal{F}^T} = \int_0^T \frac{\sin \sqrt{z}(T-s)}{\sqrt{z}} g(s) ds,$$

which proves the statement.  $\square$

We notice that initially the Krein equations were derived using purely dynamical approach (see [2, 6]).

So, having constructed reproducing kernel  $J_z(\lambda)$  from Krein equation (3.24) and convolution formula (3.23), we can recover  $E(\lambda)$  using Theorem 1, all condition of which are clearly satisfied.

We show that the fact that  $E(\lambda)$  is a Hermite-Biehler function follows from the positivity of  $C^T$ . Indeed, as it follows from (3.4),

$$\begin{aligned} \frac{|E(z)|^2 - |E(\bar{z})|^2}{4 \operatorname{Im} z} &= J_z(z) = \int_0^T \frac{\sin \sqrt{z}(T-s)}{\sqrt{z}} j_z(s) ds \\ &= \int_0^T \frac{\sin \sqrt{z}(T-s)}{\sqrt{z}} \left( (C^T)^{-1} \frac{\sin \sqrt{z}(T-\cdot)}{\sqrt{z}} \right) (s) ds \\ &= \left( \left( (C^T)^{-1} \right)^* \frac{\sin \sqrt{z}(T-s)}{\sqrt{z}}, \frac{\sin \sqrt{z}(T-s)}{\sqrt{z}} \right)_{\mathcal{F}^T} > 0, \end{aligned}$$

where the last inequality follows from the positivity of  $C^T$ .

If we know the De Branges space  $B_S^T$ , we can recover the potential  $q(x)$ ,  $x \in (0, T)$  using the general theory of canonical systems [8, 19, 17, 18], or using the Boundary Control method (we need to know the operator  $C^T$  only!). For the details see [2, 6].

#### 4. Dirac system on the half-line

We consider the operator of the Dirac system on the half-line: introduce the matrix  $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and a matrix potential  $V = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$ ,  $p, q \in C_{loc}^1(R_+)$ . We set  $\mathcal{D} := J + V$  on  $L_2(\mathbb{R}_+, \mathbb{R}^2) \ni \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$  with Dirichlet condition  $\Phi_1(0) = 0$ .

Let  $\theta(x, z) = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$  be the solution to the following Cauchy problem

$$(4.1) \quad \begin{cases} J\theta_x + V\theta = z\theta, & x > 0, \\ \theta_1(0, z) = 0, & \theta_2(0, z) = 1. \end{cases}$$

We fix  $N \in \mathbb{R}_+$  and show that  $E(z) := \theta^1(N, z) - i\theta^2(N, z)$  is a Hermite-Biehler function. Let us evaluate (2.1), counting that  $\overline{\theta(x, z)} = \theta(\bar{z})$ :

$$\begin{aligned} J_z(\xi) &= \frac{\left(\overline{\theta^1(z)} + i\overline{\theta^2(z)}\right) (\theta^1(\xi) - i\theta^2(\xi)) - \left(\overline{\theta^1(z)} - i\overline{\theta^2(z)}\right) (\theta^1(\xi) + i\theta^2(\xi))}{2i(\bar{z} - \xi)} \\ (4.2) \qquad \qquad \qquad &= \frac{\overline{\theta^2(z)}\theta^1(\xi) - \overline{\theta^1(z)}\theta^2(\xi)}{\bar{z} - \xi} \end{aligned}$$

We take points  $z, \xi \in \mathbb{C}$  and consider

$$\begin{aligned} J\overline{\theta'}(x, z) + V(x)\overline{\theta(x, z)} &= \bar{z}\overline{\theta(x, z)}, \\ J\theta'(x, \xi) + V(x)\theta(x, \xi) &= \xi\theta(x, \xi). \end{aligned}$$

multiply the first equation by  $\theta(\xi)$ , multiply the second by  $\bar{\theta}(z)$  and subtract from the first to get

$$\left(J\overline{\theta'}(z), \theta(\xi)\right)_{\mathbb{R}^2} - \left(J\overline{\theta'}(\xi), \overline{\theta(z)}\right)_{\mathbb{R}^2} = (\bar{z} - \xi) (\overline{\theta(z)}, \theta(\xi))_{\mathbb{R}^2}.$$

We integrate the latter equality from zero to  $N$  and evaluate:

$$\begin{aligned} (\bar{z} - \xi) \int_0^N \left(\overline{\theta(z)}, \theta(\xi)\right)_{\mathbb{R}^2} dx &= \int_0^N \left[ \left(\overline{\theta'_2(z)}\theta_1(\xi) - \overline{\theta'_1(z)}\theta_2(\xi)\right) - \right. \\ &\quad \left. - \left(\overline{\theta'_2(\xi)}\theta_1(z) - \overline{\theta'_1(\xi)}\theta_2(z)\right) \right] dx = \overline{\theta_2(z)}\theta_1(\xi) - \overline{\theta_1(z)}\theta_2(\xi)|_{x=N}. \end{aligned}$$

From here (see also (4.2)) follows that

$$J_z(\xi) = \int_0^N \left(\overline{\theta(x, z)}, \theta(x, \xi)\right)_{\mathbb{R}^2} dx.$$

Taking  $\xi = z$ :

$$0 < \int_0^N |\theta(x, z)|^2 dx = J_z(z) = \frac{|E(z)|^2 - |E(\bar{z})|^2}{2i(-2 \operatorname{Im} z)} = \frac{|E(z)|^2 - |E(\bar{z})|^2}{4 \operatorname{Im} z},$$

which proves  $E$  to be a Hermite-Biehler function. By this function one can construct the De Branges space  $\widehat{B}_D^N$ . On the contrary, having in hands this space one can use the De Branges technique [8, 19] to recover the canonical system (the Dirac system or the potential matrix  $V(x)$ ) it comes from. Below we use the dynamical approach to construct the De Branges space for the Dirac system.

With a Dirac operator we associate the initial boundary-value problem

$$(4.3) \quad \begin{cases} iu_t + Ju_x + Vu = 0, & x > 0, \quad 0 < t < T, \\ u|_{t=0} = 0, & x \geq 0, \\ u_1|_{x=0} = f, & 0 \leq t \leq T, \end{cases}$$

where  $T > 0$  is a final moment;  $f = f(t)$  is a complex-valued function (*boundary control*);  $u = u^f(x, t) = \begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix}$  is a solution. We denote the outer space of (4.3), the set of controls by  $\widetilde{\mathcal{F}}^T := L_2((0, T); \mathbb{C})$  with the scalar product  $(f, g)_{\mathcal{F}^T} = \int_0^T \overline{f(t)}g(t) dx$ . In [7] the authors proved the following

**THEOREM 4.** *The solution to (4.3) admits the following representation:*

$$(4.4) \quad u^f(x, t) = f(t - x) \begin{pmatrix} 1 \\ i \end{pmatrix} + \int_x^t w(x, s) f(t - s) ds, \quad x \geq 0, \quad 0 \leq t \leq T$$



holds with  $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$  being a vector-kernel such that  $w|_{t < x} = 0$ ,  $w|_{\Delta^T} \in C^1(\Delta^T; \mathbb{C}^2)$ , and  $w_1(0, \cdot) = 0$ .

The response operator  $R^T : \tilde{\mathcal{F}}^T \mapsto \tilde{\mathcal{F}}^T$  with the domain  $\{f \in C^2(0, T; \mathbb{C}^2) \mid f(0) = 0\}$ , the analog of dynamical Dirichlet to Neumann map is defined by

$$(4.5) \quad R^T f := u_2^f(0, t), \quad 0 < t < T.$$

from (4.4) we deduce

$$(4.6) \quad (Rf)(t) = if(t) + \int_0^t r(s)f(t-s) ds, \quad r(s) := w_2(0, s).$$

The speed of the wave propagation for (4.3) is equal to one, so the natural set up of the dynamical inverse problem is to recover  $V(x)$ ,  $x \in (0, T)$  from  $R^{2T}$ , or what is equivalent, from  $r(t)$ ,  $t \in (0, 2T)$ .

For the vector functions  $f, g \in L_2(\mathbb{R}_+, \mathbb{R}^2)$  define the Fourier transform (see [13])

$$(4.7) \quad \begin{aligned} \left( F \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (\lambda) &= F(\lambda) = \int_0^\infty f_1(x)\theta_1(x, \lambda) + f_2(x)\theta_2(x, \lambda) dx, \\ \left( F \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right) (\lambda) &= G(\lambda) = \int_0^\infty g_1(x)\theta_1(x, \lambda) + g_2(x)\theta_2(x, \lambda) dx \end{aligned}$$

Then there exist the measure  $d\rho(\lambda)$  such that

$$\begin{aligned} f_1(x) &= \int_{-\infty}^\infty F(\lambda)\theta_1(x, \lambda) d\rho(\lambda), \quad f_2(x) = \int_{-\infty}^\infty F(\lambda)\theta_2(x, \lambda) d\rho(\lambda), \\ g_1(x) &= \int_{-\infty}^\infty G(\lambda)\theta_1(x, \lambda) d\rho(\lambda), \quad g_2(x) = \int_{-\infty}^\infty G(\lambda)\theta_2(x, \lambda) d\rho(\lambda), \end{aligned}$$

and Parseval identity holds

$$\begin{aligned} \int_0^\infty f_1^2(x) + f_2^2(x) dx &= \int_{-\infty}^\infty F^2(\lambda) d\rho(\lambda), \\ \int_0^\infty f_1(x)g_1(x) + f_2(x)g_2(x) dx &= \int_{-\infty}^\infty F(\lambda)G(\lambda) d\rho(\lambda). \end{aligned}$$

The solution to (4.3) admits the spectral representation:

$$(4.8) \quad u^f(x, t) = \int_{-\infty}^\infty \int_0^t e^{i\lambda s} if(t-s) ds \theta(x, \lambda) d\rho(\lambda).$$

We denote the set of states by  $\mathcal{H}^T := L_2((0, T); \mathbb{C}^2)$  with an inner product  $(a, b)_{\mathcal{H}^T} := \int_{\Omega^T} \overline{a(x)} \cdot b(x) dx$ , it is the *inner space* of the system (4.3). Thus for all  $T > 0$ ,  $u^f(\cdot, T) \in \mathcal{H}^T$ .

We define the *control operator*  $\widetilde{W}^T : \tilde{\mathcal{F}}^T \mapsto \mathcal{H}^T$  by  $\widetilde{W}^T f := u^f(\cdot, T)$  and observe (see [7]) that is not an isometry, as it easily follows from (4.4),  $\widetilde{W}^T \tilde{\mathcal{F}}^T \neq \mathcal{H}^T$ . To "improve" the lack of the controllability, we consider the auxiliary system

$$(4.9) \quad \begin{cases} iv_t - Jv_x - Vv = 0, & 0 < x < T, \quad 0 < t < T \\ v|_{t=0} = 0 \\ v_1|_{x=0} = g, & 0 \leq t \leq T \end{cases}$$

The solution  $v = v^g(x, t)$  are connected with the solutions to (4.3) by

$$v^g(x, t) = \overline{u^g(x, t)}.$$

Then we introduce the extended set of controls  $\mathcal{F}^T := L_2((0, T); \mathbb{C}^2)$ , and as a (extended) state of Dirac system at the time  $t = T$  we put  $u^f(\cdot, T) + v^g(\cdot, T)$ . The new "extended" control operator we define by  $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ ,

$$(4.10) \quad W^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := u^{f_1}(\cdot, T) + v^{f_2}(\cdot, T).$$

The following statement is proved in [7]:

**THEOREM 5.** *The "extended" control operator  $W^T$  is an isomorphism between  $\mathcal{F}^T$  and  $\mathcal{H}^T$ .*

The spectral representation of  $v^g$  is

$$(4.11) \quad v^g(x, t) = \int_{-\infty}^{\infty} \int_0^t e^{-i\lambda s} (-i)g(t-s) ds \theta(x, \lambda) d\rho(\lambda).$$

Taking the the Fourier transform (4.7) of  $u^f(\cdot, T)$  and  $v^g(\cdot, T)$  for  $\lambda \in \mathbb{R}$  we get respectively:

$$(4.12) \quad (Fu^f(\cdot, T))(\lambda) = \int_0^T e^{i\lambda s} i f(T-s) ds,$$

$$(4.13) \quad (Fv^g(\cdot, T))(\lambda) = - \int_0^T e^{-i\lambda s} i g(T-s) ds.$$

The *connecting operator*  $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  is defined by the quadratic form

$$\left( C^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{F}^T} = \left( W^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{H}^T}.$$

Notice that  $C^T$  is positive isomorphism in  $\mathcal{F}^T$ , see [7].

We can evaluate making use of (4.8), (4.11), (4.12), (4.13) and Parseval identity:

$$(4.14) \quad \begin{aligned} \left( C^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{F}^T} &= \int_{-\infty}^{\infty} \overline{\left( FW^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)} \left( FW^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right) d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \int_0^T \overline{\begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \int_0^T \begin{pmatrix} ie^{i\lambda(T-t)} \\ -ie^{-i\lambda(T-t)} \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} ds d\rho(\lambda). \end{aligned}$$

The important fact proved in [7] that  $C^T$  admits a representation in terms of inverse data:

**THEOREM 6.** *The control operator is represented in terms of inverse dynamical data:*

$$(4.15) \quad (C^T a)(t) = 2a(t) + \int_0^T c^T(t, s)a(s) ds, \quad 0 \leq t \leq T$$

where  $c^T(t, s)$  is a matrix kernel with the elements

$$(4.16) \quad \begin{aligned} c_{11}(t, s)(t) &= -i[r(t-s) - \bar{r}(s-t)], & c_{12}(t, s) &= -i\bar{r}(2T-t-s), \\ c_{21}(t, s) &= i r(2T-t-s), & c_{22}(t, s)(t) &= i[\bar{r}(t-s) - r(s-t)], \end{aligned}$$

and in terms of inverse spectral data:

$$(4.17) \quad \left( C^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (x) = \int_0^T \left( C(x, y) \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \right) dy$$

where the generalized kernel of  $C^T$  is given by

$$C(t, s) = \int_{-\infty}^{\infty} \overline{c(s, \lambda)} \otimes c^T(t, \lambda) d\rho(\lambda), \quad c(s, \lambda) = \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}.$$

and action is given by the r.h.s. of (4.14).

Since  $(Fu^f(\cdot, T))(\lambda)$ ,  $(Fv^g(\cdot, T))(\lambda)$  are analytic in  $\mathbb{C}$ , and on real line are given by (4.12), (4.13) it follows that the Fourier transform of "extended" state at time  $t = T$  can be analytically continued on  $\mathbb{C}$  by the formula

$$(4.18) \quad \left( FW^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (\lambda) = \int_0^T \left( i \begin{pmatrix} e^{i\lambda s} \\ -e^{-i\lambda s} \end{pmatrix}, \begin{pmatrix} f_1(T-s) \\ f_2(T-s) \end{pmatrix} \right) ds, \quad \lambda \in \mathbb{C}.$$

We introduce the Be Branges space associated to Dirac system as a set of Fourier transforms of the (extended) states of the system (4.3) at the moment  $T$ :

$$B_D^T := \left\{ F(\lambda) = \left( FW^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (\lambda) \mid \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{F}^T \right\}$$

The relation (4.18) implies that

$$(4.19) \quad B_D^T = \left\{ \int_0^T \left( i \begin{pmatrix} e^{i\lambda s} \\ -e^{-i\lambda s} \end{pmatrix}, \begin{pmatrix} f_1(T-s) \\ f_2(T-s) \end{pmatrix} \right) ds \mid \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{F}^T \right\}$$

In  $B_D^T$  we introduce the scalar product by

$$[F, G]_{B_D^T} := \left( C^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{F}^T}, \quad F, G \in B_D^T.$$

According to (4.14):

$$\begin{aligned} [F, G]_{B_D^T} &:= \int_{-\infty}^{\infty} \overline{\int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}, \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \right) ds} \int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-t)} \\ -ie^{-i\lambda(T-t)} \end{pmatrix}, \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \right) ds d\rho(\lambda) \\ F(\lambda) &= \int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}, \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \right) ds, \quad G(\lambda) = \int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-t)} \\ -ie^{-i\lambda(T-t)} \end{pmatrix}, \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \right) ds. \end{aligned}$$

Since  $C^T$  is a positive isomorphism in  $\mathcal{F}^T$ , the space  $B_D^T$  with the norm generated by  $[\cdot, \cdot]_{B_D^T}$  is a Hilbert space. Let  $J_z(\lambda)$  be the reproducing kernel in  $B_D^T$ , the latter means that

$$(4.20) \quad [J_z, F]_{B_D^T} = F(z), \quad \forall F \in B_D^T.$$

We will look for  $J_z$  in the form:

$$(4.21) \quad J_z(\lambda) = \int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}, \begin{pmatrix} j_1^z(s) \\ j_2^z(s) \end{pmatrix} \right) ds,$$

then from (4.20) and definition of the scalar product we deduce

$$[J_z, F]_{B_D^T} = \left( C^T \begin{pmatrix} j_1^z \\ j_2^z \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)_{\mathcal{F}^T} = F(z) = \int_0^T \left( \begin{pmatrix} ie^{iz(T-s)} \\ -ie^{-iz(T-s)} \end{pmatrix}, \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \right) ds,$$

from where due to the arbitrariness of  $f_1, f_2$  we arrive at the following equation on  $\begin{pmatrix} j_1^z \\ j_2^z \end{pmatrix}$ :

$$(4.22) \quad C^T \begin{pmatrix} j_1^z \\ j_2^z \end{pmatrix} = \overline{\begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}}, \quad 0 \leq t \leq T.$$

We emphasize that equation (4.22) is Krein equations and can be used for solving the inverse problem of the recovering potential from the dynamical (or spectral) inverse data.

Let us show that  $\begin{pmatrix} j_1^z \\ j_2^z \end{pmatrix}$  is a solution to the following *special control problem*.

We fix  $z \in \mathbb{C}$  and consider the control problem to find  $\begin{pmatrix} f_1^z \\ f_2^z \end{pmatrix} \in \mathcal{F}^T$  such that

$$(4.23) \quad W^T \begin{pmatrix} f_1^z \\ f_2^z \end{pmatrix}(\cdot, T) = \theta(\cdot, z), \quad \text{on } (0, T).$$

Since  $W^T$  is boundedly invertible, such a control  $\begin{pmatrix} f_1^z \\ f_2^z \end{pmatrix}$  exists.

LEMMA 2. *The solution of the special control problem (4.23) can be found as a unique solution to the Krein equation (4.22).*

PROOF. We take the equality (4.23) and multiply it in  $\mathcal{H}^T$  by  $W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$  for some  $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{F}^T$ . As a result we get that

$$(4.24) \quad \left( W^T \begin{pmatrix} f_1^z \\ f_2^z \end{pmatrix}, W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{H}^T} = \left( \theta(\cdot, z), W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{H}^T} = \overline{\int_0^T W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(x) \theta(x, z) dx}.$$

The r.h.s. of (4.24) can be evaluated as (see 4.18):

$$(4.25) \quad \begin{aligned} & \overline{\int_0^T W^T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(x) \theta(x, z) dx} \\ &= \overline{\int_0^T \int_{-\infty}^{\infty} \int_0^T \left( \begin{pmatrix} ie^{i\lambda s} \\ -ie^{-i\lambda s} \end{pmatrix}, \begin{pmatrix} g_1(T-s) \\ g_2(T-s) \end{pmatrix} \right) ds \theta(x, \lambda) d\rho(\lambda) \theta(x, z) dx} \\ &= \int_0^T \left( i \begin{pmatrix} e^{i\lambda(T-s)} \\ -e^{-i\lambda(T-s)} \end{pmatrix}, \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} \right) ds, \end{aligned}$$

From (4.24), (4.25) we get the desired equation 4.22.  $\square$

After we found the reproducing kernel  $J_z(\lambda)$  from (4.22), (4.21), we can recover  $E(\lambda)$  making use of Theorem 1.

We show that the  $E(\lambda)$  will be the Hermite-Biehler function. It follows from (3.4),

$$\begin{aligned} \frac{|E(z)|^2 - |E(\bar{z})|^2}{4 \operatorname{Im} z} &= J_z(z) = \int_0^T \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix} \begin{pmatrix} j_1^z(s) \\ j_2^z(s) \end{pmatrix} ds \\ &= \int_0^T \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix} \left( (C^T)^{-1} \overline{\begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}} \right) (s) ds \\ &= \left( \left( (C^T)^{-1} \right)^* \overline{\begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}}, \overline{\begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}} \right)_{\mathcal{F}^T} > 0, \end{aligned}$$

where the last inequality follows from the positivity of  $C^T$ .

For positive  $N$  we can consider the Dirac system on  $(0, N)$  with some self-adjoint boundary condition at  $x = N$  :

$$(4.26) \quad \begin{cases} JU_x + VU = zU, & 0 < x < N, \\ U_1(0, z) = 0, & \alpha U_1(N, z) + \beta U_2(N, z) = 0. \end{cases}$$

The (discrete) measure corresponding to (3.19) we denote by  $d\rho_N(\lambda)$ .

REMARK 2. *Due to the finite speed of wave propagation in the dynamical system (4.3), equal to one, in all formulae starting from spectral representation of the solution (4.8), we can substitute the measure  $d\rho(\lambda)$  by any measure  $d\rho_N(\lambda)$  with  $N \geq T$ . In particular*

$$[F, G]_{B_D^T} = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho(\mu) = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho_N(\mu)$$

If we know the De Branges space  $B_D^T$ , we can recover the canonical system connected with this space using the De Branges theory [8, 19], or recover the Dirac system (the matrix potential  $V$ ) using the Boundary Control method. For the details see [7].

**4.1. Special case: connection between Dirac and Schrödinger De Branges spaces.** We consider the system (4.3) with the special matrix potential

$$(4.27) \quad V = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \quad q \text{ is differentiable, } q(0) = 0.$$

We differentiate (4.3) w.r.t.  $t$  and  $x$  to get

$$\begin{aligned} iu_{tt}^1 + u_{xt}^2 + qu_t^2 &= 0, \\ iu_{tt}^2 - u_{xt}^1 + qu_t^1 &= 0, \\ iu_{tx}^2 - u_{xx}^1 + (qu^1)_x &= 0 \end{aligned}$$

On introducing the special potential

$$(4.28) \quad Q(x) = q_x(x) + q^2(x),$$

it is easy to see that  $u^1$  satisfies the wave equation with this potential:

$$(4.29) \quad u_{tt}^1 - u_{xx}^1 + Q(x)u^1 = 0, \quad x \geq 0, t \geq 0.$$

Taking into account initial condition in (4.3) and the equation  $iu_t^1 + u_x^2 + qu^2 = 0$  at  $t = 0$ , we arrive at the initial conditions

$$(4.30) \quad u^1(x, 0) = u_t^1(x, 0) = 0.$$

Counting the last equality in (4.3), we get the boundary condition

$$(4.31) \quad u^1(0, t) = f(t).$$

Denote by  $R_S$  the response operator (3.9), (3.10) for the wave equation (4.29), (4.30), (4.31), and by  $R_D$  the response operator (4.5) (4.6) for the Dirac systems (4.3) with the matrix potential (4.27). Everywhere below the subscripts  $S$  and  $D$  being used refer the object to the Schrödinger or Dirac system. For  $R_S$  and  $R_D$  we have by (3.10) and (4.6):

$$\begin{aligned} (R_S^f)(t) &= (r_S * f)(t), \quad r_S(t) = -\delta'(t) + \tilde{r}_S(t), \\ (R_D f)(t) &= (r_D * f)(t), \quad r_D(t) = i\delta(t) + \tilde{r}_D(t), \end{aligned}$$

here we separate singular and regular parts in integral kernels. On the other hand, we can obtain from the Dirac system at  $t = 0$ :

$$u_x^1(0, t) = iu_t^2(0, t) + q(0)u^1(0, t) = i(R_D f)_t(t) + q(0)f(t) = i((R_D f)(t))'.$$

Thus for arbitrary  $f \in C_0^\infty(0, +\infty)$  we have that

$$(R_S f)(t) = (r_S * f)(t) = i(R_D f)'(t) = i(r_D' * f)(t).$$

The latter leads to the following relation between the kernels of the response operators:

$$(4.32) \quad r_S(t) = i r_D'(t).$$

The spectral representations (3.11) and (4.8) implies

$$(4.33) \quad (R_S f)(t) = u_x^f(0, t) = \int_{-\infty}^{\infty} \int_0^t \frac{\sin \sqrt{\lambda} s}{\sqrt{\lambda}} f(t-s) ds d\rho_S(\lambda),$$

$$(4.34) \quad (R_D f)(t) = u^2(0, t) = \int_{-\infty}^{\infty} \int_0^t e^{i\lambda s} i f(t-s) ds d\rho_D(\lambda).$$

Then from (4.32), (4.33), (4.34) follows the equality of the generalized kernels of the response functions (see [2, 15]):

$$\int_{-\infty}^{\infty} -i\lambda e^{i\lambda t} d\rho_D(\lambda) = \int_{-\infty}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d\rho_S(\lambda)$$

equating the real parts (the imaginary part in the l.h.s have to be equal to zero), we get

$$\int_{-\infty}^{\infty} \lambda \sin \lambda t d\rho_D(\lambda) = \int_{-\infty}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d\rho_S(\lambda).$$

The latter equality yields

$$(4.35) \quad \rho_S(\lambda) = \int_0^{\sqrt{\lambda}} \alpha^2 d\rho_D(\alpha).$$

How the De Branges spaces of Schrödinger and Dirac operators are connected in our special situation? The De Branges spaces  $B_D^T$ ,  $B_S^T$  corresponding Dirac and Schrödinger systems consist of functions of the type (see (4.19), (3.14)):

$$\begin{aligned} F(\lambda) &= \int_0^T \left( i \begin{pmatrix} e^{i\lambda s} \\ -e^{-i\lambda s} \end{pmatrix}, \begin{pmatrix} f(T-s) \\ g(T-s) \end{pmatrix} \right) ds, \quad f, g \in L_2((0, T); \mathbb{C}), \\ G(\mu) &= \int_0^T \frac{\sin \sqrt{\mu} s}{\sqrt{\mu}} h(T-s) ds, \quad h \in L_2(0, T). \end{aligned}$$

Consider the subspace  $B_s^T \subset B_D^T$ , generated by the vector functions of the special type:  $-\frac{1}{2} \begin{pmatrix} f \\ f \end{pmatrix}$  with real-valued  $f \in L_2((0, T); \mathbb{R})$ . In this case

$$(4.36) \quad B_s^T := \left\{ \int_0^T \sin \lambda s f(T-s) ds \mid f \in L_2(0, T) \right\}.$$

We take  $F \in B_s^T$  and evaluate the norm:

$$\begin{aligned} [F, F]_{B_D^T} &= \frac{1}{4} \left( C_D^T \begin{pmatrix} f \\ f \end{pmatrix}, \begin{pmatrix} f \\ f \end{pmatrix} \right)_{\mathcal{F}_D^T} \\ &= \frac{1}{4} \int_{-\infty}^{\infty} \overline{\int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-s)} \\ -ie^{-i\lambda(T-s)} \end{pmatrix}, \begin{pmatrix} f(s) \\ f(s) \end{pmatrix} \right) ds} \int_0^T \left( \begin{pmatrix} ie^{i\lambda(T-t)} \\ -ie^{-i\lambda(T-t)} \end{pmatrix}, \begin{pmatrix} f(t) \\ f(t) \end{pmatrix} \right) ds d\rho_D(\lambda) \\ &= \int_{-\infty}^{\infty} \int_0^T \int_0^T \sin \lambda s f(T-s) \sin \lambda t f(T-t) dt ds d\rho_D(\lambda) \\ &= \int_{-\infty}^{\infty} \int_0^T \int_0^T \frac{\sin \sqrt{\lambda} s}{\sqrt{\lambda}} f(T-s) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} f(T-t) dt ds d\rho_S(\lambda) = (C_S^T f, f)_{\mathcal{F}_S^T} \\ &= [\tilde{F}, \tilde{F}]_{B_S^T}, \quad \tilde{F}(\lambda) = \int_0^T \frac{\sin \sqrt{\lambda} s}{\sqrt{\lambda}} f(T-s) ds. \end{aligned}$$

Thus the Schrödinger De Branges  $B_S^T$  of the system with the potential (4.28) are isometrically embedded into Dirac De Branges space  $B_D^T$  of the system with the matrix potential (4.27) and  $B_S^T$  is isometrically isomorphic to the subspace  $B_s^T$  of  $B_D^T$  generated by the functions of the special type (4.36).

## 5. Discrete Schrödinger operator

For the real sequence  $(b_n)$  we consider the discrete Schrödinger operator in  $l^2$  given by

$$(5.1) \quad \begin{cases} (H\phi)_n = \phi_{i+1} + \phi_{i-1} + b_n \phi_i, & n \geq 1, \\ (H\phi)_0 = b_1 \phi_0 + \phi_1. \end{cases}$$

Let  $\varphi$  be the the solution to

$$(5.2) \quad \begin{cases} \varphi_{i+1} + \varphi_{i-1} + b_n \varphi_i = z \varphi_i, \\ \varphi_0 = 0, \quad \varphi_1 = 1. \end{cases}$$

We fix some  $N \in \mathbb{N}$  and introduce the function  $E(z) := \varphi_N(z) - i\varphi_{N+1}(z)$  and show that it is a Hermite-Biehler function. First we observe that  $\varphi_i(\bar{z}) = \overline{\varphi_i(z)}$ . Then evaluating  $J_z$  in accordance with (2.1):

$$\begin{aligned} J_z(\xi) &= \frac{\left( \overline{\varphi_N(z)} + i\overline{\varphi_{N+1}(z)} \right) (\varphi_N(\xi) - i\varphi_{N+1}(\xi)) - \left( \overline{\varphi_N(z)} - i\overline{\varphi_{N+1}(z)} \right) (\varphi_N(\xi) + i\varphi_{N+1}(\xi))}{2i(\bar{z} - \xi)} \\ (5.3) \quad &= \frac{\left( \overline{\varphi_{N+1}(z)} \varphi_N(\xi) - \varphi_{N+1}(\xi) \overline{\varphi_N(z)} \right)}{\bar{z} - \xi}. \end{aligned}$$

Let us consider the equations

$$\begin{aligned} \overline{\varphi_{i+1}(z)} + \overline{\varphi_{i-1}(z)} + b_n \overline{\varphi_i(z)} &= \bar{z} \overline{\varphi_i(z)}, \\ \varphi_{i+1}(\xi) + \varphi_{i-1}(\xi) + b_n \varphi_i(\xi) &= \xi \varphi_i(\xi). \end{aligned}$$

On multiplying first equation by  $\varphi_i(\xi)$ , second equation by  $\overline{\varphi_i(z)}$  and subtracting second from first, we get

$$\left(\overline{\varphi_{i+1}(z)} + \overline{\varphi_{i-1}(z)}\right) \varphi_i(\xi) - (\varphi_{i+1}(\xi) + \varphi_{i-1}(\xi)) \overline{\varphi_i(z)} = (\bar{z} - \xi) \overline{\varphi_i(z)} \varphi_i(\xi).$$

Summing up left and right hand sides of the previous equality from 1 to  $N$ , we get:

$$(5.4) \quad (\bar{z} - \xi) \sum_{i=1}^N \overline{\varphi_i(z)} \varphi_i(\xi) = \varphi_{N+1}(z) \varphi_N(\xi) - \varphi_{N+1}(\xi) \overline{\varphi_N(z)}.$$

Then from (5.3), (5.4) we see that

$$J_z(\xi) = \sum_{i=1}^N \overline{\varphi_i(z)} \varphi_i(\xi),$$

and setting here  $z = \xi$  we obtain

$$0 < \sum_{i=1}^N |\varphi_i(z)|^2 = J_z(z) = \frac{|E(z)|^2 - |E(\bar{z})|^2}{4\Im z}.$$

So  $E$  is a Hermite-Biehler function. We can define De Branges space  $\widehat{B}_J^N$  based on this function. The opposite is also true: if we have a De Branges space which comes from discrete Schrödinger operator, one can recover corresponding canonical system [19] by general technique [8].

For the same sequence  $(b_n)$  we consider the dynamical system with discrete time which is a natural analog of dynamical systems governed by the wave equation with potential on a semi-axis:

$$(5.5) \quad \begin{cases} u_{n,t+1} + u_{n,t-1} - u_{n+1,t} - u_{n-1,t} - b_n u_{n,t} = 0, & n, t \in \mathbb{N}_0, \\ u_{n,-1} = u_{n,0} = 0, & n \in \mathbb{N}, \\ u_{0,t} = f_t, & t \in \mathbb{N}_0. \end{cases}$$

By analogy with continuous problems, we treat the complex sequence  $f = (f_0, f_1, \dots) \in \mathbb{C}^\infty$  as a *boundary control*. The solution to (5.5) we denote by  $u_{n,t}^f$ . In [14, 16] the following representation have been proved:

**THEOREM 7.** *The solution to (5.5) admits the following representation*

$$(5.6) \quad u_{n,t}^f = \prod_{k=0}^{n-1} f_{t-k} + \sum_{s=n}^{t-1} w_{n,s} f_{t-s-1}, \quad n, t \in \mathbb{N}_0.$$

where  $w_{n,s}$  satisfies the Goursat problem

$$(5.7) \quad \begin{cases} w_{n,t+1} + w_{n,t-1} - w_{n+1,t} - w_{n-1,t} + b_n w_{n,t} = 0, & n, s \in \mathbb{N}_0, s > n, \\ w_{n,n} = -\sum_{k=1}^n b_k, & n \in \mathbb{N}, \\ w_{0,t} = 0, & t \in \mathbb{N}_0. \end{cases}$$

**DEFINITION 1.** *For  $a, b \in l^\infty$  we define the convolution  $c = a * b \in l^\infty$  by the formula*

$$c_t = \sum_{s=0}^t a_s b_{t-s}, \quad t \in \mathbb{N}$$



By  $\mathcal{F}^T$  we denote the *outer space*, the space of controls:  $\mathcal{F}^T := \mathbb{C}^T$ ,  $f, g \in \mathcal{F}^T$ ,  $f = (f_0, \dots, f_{T-1})$  with the inner product  $(f, g)_{\mathcal{F}^T} = \sum_{k=0}^{T-1} \overline{f_k} g_k$ . As a dynamical inverse data for (5.5) we use the *response operator* which is a dynamical Dirichlet-to-Neumann map:  $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  is defined by the rule

$$(R^T f)_t = u_{1,t}^f, \quad t = 1, \dots, T.$$

By (5.6):

$$(5.8) \quad (R^T f)_t = u_{1,t}^f = a_0 f_{t-1} + \sum_{s=1}^{t-1} w_{1,s} f_{t-1-s} \quad t = 1, \dots, T.$$

$$(R^T f) = r * f_{-1}.$$

where the *response vector* is the convolution kernel of the response operator,  $r = (1, r_1, \dots, r_{T-1}) = (1, w_{1,1}, w_{1,2}, \dots, w_{1,T-1})$ .

We introduce the *inner space*, the space of states of the dynamical system (5.5)  $\mathcal{H}^T := \mathbb{C}^T$ ,  $h, l \in \mathcal{H}^T$ ,  $h = (h_1, \dots, h_T)$  with the inner product  $(h, l)_{\mathcal{H}^T} = \sum_{k=1}^T \overline{h_k} l_k$ . The *control operator*  $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$  is defined by the rule

$$W^T f := u_{n,T}^f, \quad n = 1, \dots, T.$$

We notice that in [14, 16] the authors used the real inner space (and, consequently, the real outer space), but all the results are valid for the complex controls as well. Everywhere below, unless it is mentioned, we use the real outer and inner spaces  $\mathcal{F}^T$ ,  $\mathcal{H}^T$ . In [14] the authors proved

**THEOREM 8.** *The control operator  $W^T$  is an isomorphism between  $\mathcal{F}^T$  and  $\mathcal{H}^T$ .*

According to [1, 4] there exist the spectral measure  $d\rho(\lambda)$  corresponding to (5.1) with Dirichlet condition  $\phi_0 = 0$  such that for  $u \in l^2$  the Fourier transform  $F : l^2 \mapsto L_2(\mathbb{R}, d\rho)$  is defined as

$$(5.9) \quad (Fu)(\lambda) = \sum_{n=0}^{\infty} u_n \varphi_n(\lambda)$$

and the Parseval identity holds:

$$(5.10) \quad \sum_{k=0}^{\infty} u_k v_k = \int_{-\infty}^{\infty} (Fu)(\lambda) (Fv)(\lambda) d\rho(\lambda).$$

where

$$u_k = \int_{-\infty}^{\infty} (Fu)(\lambda) \varphi_k(\lambda) d\rho(\lambda)$$

Introduce the functions

$$\begin{cases} \mathcal{T}_{t+1} + \mathcal{T}_{t-1} - \lambda_k \mathcal{T}_t = 0, \\ \mathcal{T}_0 = 0, \mathcal{T}_1 = 1. \end{cases}$$

So  $\mathcal{T}_k(2\lambda)$  are Chebyshev polynomials of the second kind. In [14, 16] the following spectral representation for the solution to (5.5) have been derived:

$$(5.11) \quad u_{n,t}^f = \int_{-\infty}^{\infty} \sum_{k=1}^t \mathcal{T}_k(\lambda) f_{t-k} \varphi_n(\lambda) d\rho(\lambda)$$

We put the following definition of the De Branges space, associated with (5.1)

$$B_J^T := \left\{ \left( F u_{\cdot, T}^f \right) (\lambda) \mid f \in \mathcal{F}^T \right\}.$$

We take  $t = T$  in (5.11) and go over the Fourier transform (5.9). For real  $\lambda$  we evaluate:

$$\left( F u_{\cdot, t}^f \right) (\lambda) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \sum_{k=1}^T \mathcal{T}_k(z) f_{T-k} \varphi_n(z) d\rho(z) \varphi_k(\lambda) = \sum_{k=1}^T \mathcal{T}_k(\lambda) f_{T-k}.$$

Notice that for  $\lambda \in \mathbb{C}$  we have the same formula due to the analyticity of the l.h.s. Thus we get the following representation for  $B_J^T$ :

$$(5.12) \quad B_J^T := \left\{ \sum_{k=1}^T \mathcal{T}_k(\lambda) f_{T-k} \mid f \in \mathcal{F}^T \right\}.$$

The *connecting operator*  $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  for (5.5) is introduced via the quadratic form:

$$(5.13) \quad (C^T f, g)_{\mathcal{F}^T} = \left( u_{\cdot, T}^f, u_{\cdot, T}^g \right)_{\mathcal{H}^T} = (W^T f, W^T g)_{\mathcal{H}^T}, \quad C^T = (W^T)^* W^T.$$

The fact that  $C^T$  can be expressed in terms of the inverse data is crucial in BC-method. The following theorem have been proved in [14]:

**THEOREM 9.** *Connecting operator admits the representation in terms of dynamical (response vector  $r$ ) inverse data*

$$(5.14) \quad C^T = C_{ij}^T, \quad C_{ij}^T = \sum_{k=0}^{T-\max i, j} r_{|i-j|+2k}, \quad r_0 = 1.$$

$$C^T = \begin{pmatrix} r_0 + r_2 + \dots + r_{2T-2} & r_1 + r_3 + \dots + r_{2T-3} & \dots & r_T + r_{T-2} & r_{T-1} \\ r_1 + r_3 + \dots + r_{2T-3} & r_0 + r_2 + \dots + r_{2T-4} & \dots & \dots & r_{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{T-3} + r_{T-1} + r_{T+1} & \dots & r_0 + r_2 + r_4 & r_1 + r_3 & r_2 \\ r_T + r_{T-2} & \dots & r_1 + r_3 & 1 + r_2 & r_1 \\ r_{T-1} & r_{T-2} & \dots & r_1 & r_0 \end{pmatrix}$$

and spectral (spectral measure  $d\rho$ ) inverse data:

$$(5.15) \quad C_{l+1, m+1}^T = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda) \mathcal{T}_{T-m}(\lambda) d\rho(\lambda), \quad l, m = 0, \dots, T-1.$$

and

$$r_{k-1} = \int_{-\infty}^{\infty} \mathcal{T}_k(\lambda) d\rho(\lambda), \quad k \in \mathbb{N}.$$

In  $B_J^T$  we introduce the scalar product by

$$(5.16) \quad [F, G]_{B_J^T} = (C^T f, g)_{\mathcal{F}^T}.$$

Since  $C^T$  is a positive isomorphism, the space  $B_J^T$  equipped with the norm generated by (5.16) is a Hilbert space. We evaluate (5.16) using (5.10):

$$[F, G]_{B_J^T} = \left( u_{\cdot, T}^f, u_{\cdot, T}^g \right)_{\mathcal{H}^T} = \int_{-\infty}^{\infty} \overline{(F u_{\cdot, T}^f)(\lambda)} (F u_{\cdot, T}^g)(\lambda) d\rho(\lambda) = \int_{-\infty}^{\infty} \overline{F(\lambda)} G(\lambda) d\rho(\lambda).$$

We will be looking for the reproducing kernel in  $B_J^T$  in the form

$$(5.17) \quad J_z(\lambda) = \sum_{k=1}^T \mathcal{T}_k(\lambda) j_{T-k}^z,$$

then by definition we should have for all  $F \in B_J^T$  that  $[J_z, F]_{B_J^T} = F(z)$ . The latter immediately implies that for  $z \in \mathbb{C}$

$$[J_z, F]_{B_J^T} = (C^T j^z, f)_{\mathcal{F}^T} = F(z) = \sum_{k=1}^T \mathcal{T}_k(z) f_{T-k} = \left( \begin{pmatrix} \overline{\mathcal{T}_T(z)} \\ \mathcal{T}_{T-1}(z) \\ \vdots \\ \mathcal{T}_1(z) \end{pmatrix}, \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{T-1} \end{pmatrix} \right)_{\mathcal{F}^T}.$$

From where we get the following equation on  $j^z$ :

$$(5.18) \quad C^T j^z = \begin{pmatrix} \overline{\mathcal{T}_T(z)} \\ \mathcal{T}_{T-1}(z) \\ \vdots \\ \mathcal{T}_1(z) \end{pmatrix}.$$

We set up the *special control problem*: for  $z \in \mathbb{C}$  to find  $j_z \in \mathcal{F}^T$  (specifically at this point we need complex controls!) such that

$$(5.19) \quad (W^T j^z)_n = \varphi_n(z), \quad n = 1, \dots, T.$$

LEMMA 3. *The solution to the special control problem can be found as a solution to (5.18).*

PROOF. We multiply (5.19) by  $W^T g$ ,  $g \in \mathcal{F}^T$  in  $\mathcal{H}^T$ . As result we get that

$$(5.20) \quad (C^T j^z, g)_{\mathcal{F}^T} = (\varphi(\cdot, z), W^T g)_{\mathcal{H}^T} = \sum_{n=1}^T \overline{(W^T g)_n} \varphi_n(z).$$

We evaluate the r.h.s. of the above equality using the spectral representation (5.11):

$$(5.21) \quad \begin{aligned} \sum_{n=1}^T \overline{(W^T g)_n} \varphi_n(z) &= \overline{\sum_{n=1}^T \int_{-\infty}^{\infty} \sum_{k=1}^T \mathcal{T}_k(\lambda) g_{T-k} \varphi_n(\lambda) d\rho(\lambda) \varphi_n(z)} \\ &= \sum_{k=1}^T \mathcal{T}_k(\lambda) g_{T-k} = \left( \begin{pmatrix} \overline{\mathcal{T}_T(z)} \\ \mathcal{T}_{T-1}(z) \\ \vdots \\ \mathcal{T}_1(z) \end{pmatrix}, \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{T-1} \end{pmatrix} \right)_{\mathcal{F}^T}. \end{aligned}$$

From (5.20) and (5.21) the statement of the lemma follows.  $\square$

The positivity of  $C^T$  yields the function  $E$  to be from Hermite-Biehler class: from (5.17), (5.18) we easily get:

$$\frac{|E(z)|^2 - |E(\bar{z})|^2}{4 \operatorname{Im} z} = J_z(z) = \left( \begin{pmatrix} \overline{\mathcal{T}_T(z)} \\ \mathcal{T}_{T-1}(z) \\ \vdots \\ \mathcal{T}_1(z) \end{pmatrix}, (C^T)^{-1} \begin{pmatrix} \overline{\mathcal{T}_T(z)} \\ \mathcal{T}_{T-1}(z) \\ \vdots \\ \mathcal{T}_1(z) \end{pmatrix} \right)_{\mathcal{F}^T} > 0$$

For any positive  $N$  we can consider the discrete Schrödinger operator with some self-adjoint boundary condition at  $n = N$  :

$$(5.22) \quad \begin{cases} \varphi_{i+1} + \varphi_{i-1} + b_n \varphi_i = z \varphi_i, \\ \varphi_0 = 0, \quad \alpha \varphi_{N+1} + \beta \varphi_N = 0. \end{cases}$$

The (discrete) measure corresponding to (5.22) we denote by  $d\rho_N(\lambda)$ .

REMARK 3. *Due to the finite speed of propagation in the dynamical system (5.5), in all formulaes starting from spectral representation of the solution (5.11), we can substitute the measure  $d\rho(\lambda)$  by any measure  $d\rho_N(\lambda)$  with  $N > T$ . In particular*

$$[F, G]_{B_J^T} = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho(\mu) = \int_{-\infty}^{\infty} \overline{F(\mu)} G(\mu) d\rho_N(\mu)$$

So, having constructed reproducing kernel  $J_z$  by (5.18), (5.17), by Theorem 1 we can recover the Hermite-Biehler function  $E$ , the space  $B_J^T$  is based on. Having in hands De Branges space  $B_J^T$ , one can recover the underlying canonical system using the general approach [8], or one can use the Boundary Control method for discrete Schrodinger operator as it described in [14, 16].

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